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Improved Edelbaum's Approach to Optimize Low Earth/Geostationary Orbits Low-Thrust Transfers

Lorenzo Casalino* and Guido Colasurdo†
Politecnico di Torino, 10129 Torino, Italy

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Edelbaum's approach to the optimization of low-thrust transfers is revisited and some simplifications are removed. The variation of the spacecraft mass due to the propellant consumption is considered in the case of constant thrust, and the corresponding numerical result is compared with Edelbaum's solution. The approach is then extended to consider variable specific impulse and thrust magnitude with constant power level. The payload increment is first computed maintaining Edelbaum's suboptimal control strategy (i.e., constant-thrust direction during each half-revolution). An analytical solution of the quasi-circular one-revolution transfer is then found using the optimal control of both the thrust direction and magnitude. The very-low-thrust multirevolution problem is easily solved by assembling many one-revolution basic trajectories; in particular, the transfer from a 28.5 deg inclined low Earth orbit to the equatorial geostationary orbit is considered. Exact numerical solutions for both constant and variable specific impulse have also been obtained using an indirect optimization method: the accuracy of the solution based on the quasi-circular approximation has been verified.

I. Introduction

THE velocity increment, which can be obtained using a given propellant mass, is proportional to the specific impulse I_{sp}. Electric propulsion (EP) provides a specific impulse which can be up to 10 times larger when compared with chemical propulsion, and therefore allows great propellant savings. On the other hand, the necessity of a power generator limits the thrust level that can be provided by EP and low-thrust trajectories must be flown, whereas the large thrust of chemical propulsion allows the spacecraft to fly more efficient ballistic trajectories with impulsive velocity changes. Gravitational and misalignment losses arise when low-thrust trajectories are considered; continuous thrusting is usually adopted, but the task of optimizing the trajectory to perform a minimum-time mission (i.e., of determining the thrust direction which minimizes the propellant consumption) is usually complex. Moreover, variable I_{sp} EP systems will probably be available soon [1] and an additional control, the thrust magnitude, must be considered in the optimization problem; in fact, when the available power is given, either a large specific impulse or a large thrust level may be advisable, depending on the position of the spacecraft along its trajectory.

The acceleration, which is provided by the propulsion system, must be compared with the gravitational acceleration of the main body. In the case of interplanetary trajectories, the thrust of EP systems is usually comparable to the solar gravity (at least if the spacecraft does not move very close to the sun) and the optimal trajectories perform a limited number of revolutions around the sun. In this case, direct and indirect methods are effective in the optimization of EP transfers between the Earth and other bodies of the solar system, also in the case of complex trajectories exploiting multiple gravity assists. For instance, the authors have applied an indirect optimization method to trajectories toward inner and outer planets, and near-Earth and main-belt asteroids, either with constant or variable specific impulse [2–5].

Inside the Earth's sphere of influence, the thrust acceleration, which is provided by EP, is several orders of magnitude lower than the gravitational acceleration and a large number of revolutions around the main body must be performed. For instance, the low Earth orbit–geostationary orbit (LEO–GEO) transfer typically requires several days (more than 100) and revolutions (several hundreds). When a large number of revolutions is performed, the control variables, namely the thrust angles, exhibit oscillations during each revolution; it is extremely difficult to obtain the convergence to the exact solution, when numerical optimization methods are used, even for the simpler planar case. Direct methods require a very large number of parameters; the initial guess at the adjoint variables, and the solution of the boundary value problem, are often very difficult to obtain if indirect methods are used. Good results have sometimes been provided by recasting the problem in terms of suitable state variables [6,7] or by using approximate expressions for the adjoint variables [8,9].

During a very-low-thrust transfer from circular LEO to GEO, the eccentricity is always quite small and the inclination change, which is obtained during one revolution, is also small. The optimization of the trajectory can be carried out by assuming simplified models and searching for quasi-optimal trajectories. The solution provided by Edelbaum [10] is the classical guideline; it assumes quasi-circular orbits and constant-thrust acceleration. In particular, Edelbaum finds the steering law that minimizes the characteristic velocity ΔV to achieve an assigned circular inclined orbit after a complete revolution around the main body. The out-of-plane component of the thrust obeys a sinusoidal law with the maximum in correspondence of the nodes, where the thrust is effective in rotating the orbit plane. An elegant analytical solution for the multirevolution transfer between circular orbits with plane change (e.g., the LEO–GEO transfer) is obtained by joining many one-revolution transfers, but a constant out-of-plane thrust component must be assumed for each half-revolution. The resulting optimal share of the propulsive effort privileges the radius increment in the initial part of the maneuver, whereas the plane rotation is mainly sought in the final phases. Kechichian [11] has reformulated the problem by using the formalism of optimal control theory and the time as the independent variable, thus eliminating the ambiguities caused by double-valued functions in Edelbaum's solution.

In the present paper, the authors look again at Edelbaum's approach but some improvements are introduced, while maintaining the assumption of quasi-circular orbits. The propellant consumption

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*Associate Professor, Dipartimento di Energetica, corso Duca degli Abruzzi 24. AIAA Senior Member.

†Professor, Dipartimento di Energetica, corso Duca degli Abruzzi 24. AIAA Associate Fellow.

may be significant, in particular when the specific impulse is low; the variation of the spacecraft mass (and consequently of the thrust acceleration) is therefore first considered. Trajectories with variable thrust and specific impulse at constant power are then analyzed, as an improved management of the available power will be permitted by variable-I_{sp} thrusters. It is well known that a progressive increment of the specific impulse is convenient as the propellant is consumed and the spacecraft mass diminishes. Oscillatory variations are superimposed to this trend, because higher I_{sp} and lower thrust is preferable far from the nodes, where the spacecraft cannot efficiently use the thrust to rotate the orbit plane.

The classical constant-mass constant-I_{sp} Edelbaum's approach is presented in Sec. II. The basic one-revolution transfer between inclined orbits is analyzed in Sec. III, retaining the approximation of a quasi-circular trajectory. Edelbaum's analysis is extended to consider variable specific impulse and thrust magnitude with constant power level. In the constant-I_{sp} case, closed-form expressions for the radius and inclination increments after one revolution can only be obtained using a suboptimal control law (that is, constant-thrust angle). A closed-form solution is instead found using the optimal thrust direction and magnitude in the variable-I_{sp} case. A large number of one-revolution transfers using either the approximate or the exact control law are patched together in Sec. IV to obtain the optimal constant-I_{sp} and variable-I_{sp} transfers between circular noncoplanar orbits with a large change in radius and inclination. Numerical results concerning the transfer from a 28.5 deg inclined circular LEO to the equatorial GEO are presented and discussed in Sec. V. The solutions based on the quasi-circular approximation are also compared with exact numerical solutions that are obtained using an indirect optimization method.

II. Edelbaum's Approach to Low-Thrust Trajectories

In his famous paper [10], Edelbaum uses the classical orbital elements as the problem variables. The relevant equations [12,13] for the orbital parameters are written under the approximation of almost circular orbits (i.e., small eccentricity) and small inclination:

$$\mathrm{d}\,a/\mathrm{d}t = 2aA_v/V_0 \tag{1}$$

$$\mathrm{d}\,e/\mathrm{d}t = (2\cos\nu A_v + \sin\nu A_u)/V_0 \tag{2}$$

$$\mathrm{d}\,i/\mathrm{d}t = \cos(\omega+\nu)A_w/V_0 \tag{3}$$

$$\mathrm{d}\,\Omega/\mathrm{d}t = \sin(\omega+\nu)A_w/(iV_0) \tag{4}$$

$$\mathrm{d}\,\omega/\mathrm{d}t = [2\sin\nu A_v/e - \cos\nu A_u/e - \sin(\omega+\nu)A_w/i]/V_0 \tag{5}$$

where ν is the true anomaly.

The thrust acceleration A is assumed to be constant. It is described by three components in a local reference frame, which is based on the orbit plane: the component in the orbital plane and perpendicular to the velocity vector, toward the exterior of the orbit $A_u = A\sin\alpha\cos\beta$, the component along the velocity vector $A_v = A\cos\alpha\cos\beta$, and the component out of the orbit plane in the direction of the orbital momentum vector $A_w = A\sin\beta$. This set of equations is, however, troublesome to be handled, because of the singularities which occur when either $e = 0$ or $i = 0$. Even though Edelbaum considers low-thrust nonplanar maneuvers, and both eccentricity and inclination are small, he assumes Ω and ω as constants and neglects the corresponding equations. Actually, both Ω and ω vary during the maneuver, but Edelbaum's approach considers one complete revolution, after which Ω and ω assume again their initial values. In this case, the optimization determines $A_u = 0$ (that is, $\alpha = 0$) and the thrust angle β from

$$\tan\beta = A_w/A_v = C\cos(\omega+\nu) \tag{6}$$

where Edelbaum replaces the argument of longitude $\omega + \nu$ with the angular distance from the initial (and also final) node.

The solution of the equations of motion using the optimal control law involves elliptic integrals. Edelbaum assumes an approximate control law to obtain a closed-form solution of the single-revolution problem, which is a more suitable basis to deal with large changes in radius and inclination, which require many revolutions of the spacecraft around the main body. In fact, when β is kept constant during one revolution (to be precise, β has a constant absolute value but is kept positive during half-revolution and negative during the other half), the equations of motion can be recast to provide the change in inclination and time as a function of the velocity V, which, on a quasi-circular trajectory, is uniquely related to the orbit radius:

$$\mathrm{d}\,i/\mathrm{d}V = 2\tan\beta/(\pi V) \tag{7}$$

$$\mathrm{d}\,t/\mathrm{d}V = 1/(A\cos\beta) \tag{8}$$

A simple optimization provides the optimal value of $\sin\beta$ during each revolution, which results to be inversely proportional to the velocity: $V\sin\beta = \text{constant}$. This result is used to determine the characteristic velocity of a transfer between circular orbits with inclination change Δi, expressed by the well-known relation

$$\Delta V = \sqrt{V_0^2 - 2V_0V_f\cos(\pi\Delta i/2) + V_f^2} \tag{9}$$

III. One-Revolution Transfer

The authors look again at Edelbaum's problem but use a different set of equations. The spacecraft position (described in spherical coordinates, namely, radius r, longitude ϑ, and latitude ϕ), the velocity (described in a local reference frame by means of the radial component u, the tangential component parallel to the equatorial plane v, and the tangential component perpendicular to the equatorial plane w), and the mass m, when required, are the state variables. The true longitude ϑ (measured on the initial orbit plane, which is assumed to be the equatorial plane) is chosen as the independent variable. This section deals with the one-revolution transfer between circular orbits, which maximizes the final radius for an assigned small change of inclination. All variables are made nondimensional by assuming the radius, the corresponding circular velocity, and the spacecraft mass at the beginning of the revolution as reference values. The velocity component v is replaced by $v' = v - 1/(rv)$ to simplify the calculations.

A. Equations of Motion and Optimization

An electric propulsion system with assigned thrust power $P = Tc/2$ is used to perform the maneuver. The thrust and the effective exhaust velocity, which is proportional to the specific impulse, assume constant nominal values (T_N, c_N) when a constant-I_{sp} thruster is considered. They can be varied and c is an additional control when a variable-I_{sp} thruster is employed.

The time derivative of the state variables are

$$\mathrm{d}\,r/\mathrm{d}t = u \tag{10}$$

$$\mathrm{d}\,\vartheta/\mathrm{d}t = v/(r\cos\phi) \tag{11}$$

$$\mathrm{d}\,\phi/\mathrm{d}t = w/r \tag{12}$$

$$\mathrm{d}\,u/\mathrm{d}t = (-1/r + v^2 + w^2)/r + T_u/m \tag{13}$$

$$\mathrm{d}\,v/\mathrm{d}t = (-uv + vw\tan\phi)/r + T_v/m \tag{14}$$

$$\mathrm{d}\,w/\mathrm{d}t = -(uw - v^2\tan\phi)/r + T_w/m \tag{15}$$

$$\mathrm{d}\,m/\mathrm{d}t = -T/c \tag{16}$$

where the mass equation has been introduced; however, if a single revolution is performed, the mass variation is small because of the low-thrust assumption, and can be neglected when the thrust acceleration is computed. Therefore the thrust acceleration is $A = T = 2P/c$, as the mass is unit, and, under the assumptions of almost circular orbit and small inclination, one obtains the system of differential equations

$$\mathrm{d}\,r/\mathrm{d}\vartheta = u \tag{17}$$

$$\mathrm{d}\,u/\mathrm{d}\vartheta = v' + T_u \tag{18}$$

$$\mathrm{d}\,v'/\mathrm{d}\vartheta = -u + 2T_v \tag{19}$$

$$\mathrm{d}\,\phi/\mathrm{d}\vartheta = w \tag{20}$$

$$\mathrm{d}\,w/\mathrm{d}\vartheta = -\phi + T_w \tag{21}$$

$$\mathrm{d}\,m/\mathrm{d}\vartheta = -T/c \tag{22}$$

For low-eccentricity, low-inclination orbits, the thrust acceleration components coincide with those that have been introduced in the preceding section, and one has again $T_u = T\sin\alpha\cos\beta$, $T_v = T\cos\alpha\cos\beta$, and $T_w = T\sin\beta$.

An adjoint variable is associated to each differential equation and the Hamiltonian is defined

$$H = (\lambda_r - \lambda_v')u + \lambda_u v' + \lambda_\phi w - \lambda_w \phi + (\lambda_u \sin\alpha\cos\beta + 2\lambda_v'\cos\alpha\cos\beta + \lambda_w\sin\beta - \lambda_m/c)2P/c \tag{23}$$

The optimal thrust direction is obtained by imposing $\partial H/\partial\alpha = \partial H/\partial\beta = 0$, which provide $T_u = T\lambda_u/\Lambda$, $T_v = 2T\lambda_v'/\Lambda$, $T_w = T\lambda_w/\Lambda$, where $\Lambda = \sqrt{\lambda_u^2 + (2\lambda_v')^2 + \lambda_w^2}$ is the magnitude of the primer vector. The optimal values of the thrust angles α and β are inserted into Eq. (23) which becomes

$$H = (\lambda_r - \lambda_v')u + \lambda_u v' + \lambda_\phi w - \lambda_w\phi + (\Lambda - \lambda_m/c)2P/c \tag{24}$$

The boundary conditions at departure ($\vartheta_0 = 0$) are $r_0 = 1$, $u_0 = 0$, $v_0' = 0$, $\phi_0 = 0$, $w_0 = 0$, and $m_0 = 1$. The final radius is maximized for prescribed final inclination $\bar{i}$, assuming $\Omega_f = 0$ deg (it will be shown that the results are independent of Ω_f). The final mass is also assigned; this constraint is unnecessary when constant specific impulse is assumed (the propellant mass flow rate T_N/c_N and transfer time 2π are fixed), but it is required to obtain meaningful results when variable specific impulse is considered. Therefore, at the final point ($\vartheta_f = 2\pi$), for small final inclination and by considering that the nondimensional final velocity is unit, $u_f = 0$, $v_f' = 0$, $w_f = \bar{i}$, $\phi_f = 0$, and $m_f = \bar{m} = 1 - 2\pi T_N/c_N$.

The problem is completed by the boundary conditions for optimality which can be easily derived [14]. For radius maximization one has $\lambda_{rf} = 1$. The problem is homogeneous in the adjoint variables, which can be arbitrarily scaled to simplify the solution: $\lambda_{mf} = 1/2$ is used to replace $\lambda_{rf} = 1$.

The Euler–Lagrange equations provide the differential equations for the adjoint variables

$$\mathrm{d}\,\lambda_r/\mathrm{d}\vartheta = -\partial H/\partial r = 0 \tag{25}$$

$$\mathrm{d}\,\lambda_u/\mathrm{d}\vartheta = -\partial H/\partial u = \lambda_v' - \lambda_r \tag{26}$$

$$\mathrm{d}\,\lambda_v'/\mathrm{d}\vartheta = -\partial H/\partial v' = -\lambda_u \tag{27}$$

$$\mathrm{d}\,\lambda_\phi/\mathrm{d}\vartheta = -\partial H/\partial\phi = \lambda_w \tag{28}$$

$$\mathrm{d}\,\lambda_w/\mathrm{d}\vartheta = -\partial H/\partial w = -\lambda_\phi \tag{29}$$

$$\mathrm{d}\,\lambda_m/\mathrm{d}\vartheta = -\partial H/\partial m = 0 \tag{30}$$

which can be easily integrated to obtain

$$\lambda_r = K_1 \tag{31}$$

$$\lambda_u = K_2\sin(\vartheta - C_1) \tag{32}$$

$$\lambda_v' = K_1 + K_2\cos(\vartheta - C_1) \tag{33}$$

$$\lambda_\phi = K_3\sin(\vartheta - C_2) \tag{34}$$

$$\lambda_w = K_3\cos(\vartheta - C_2) \tag{35}$$

$$\lambda_m = 1/2 \tag{36}$$

B. Constant-Specific-Impulse Thruster

The equations of motion cannot be integrated analytically if the optimal thrust direction is adopted, and the radius and inclination change is expressed by elliptic integrals [10]. However, the integration constants, which satisfy the boundary conditions at the final point, can be determined when one complete revolution ($\vartheta_f = 2\pi$) is performed. The problem is autonomous, i.e., the independent variable ϑ does not appear in H, therefore, the Hamiltonian is constant and, in particular, $H_f = H_0$; the adjoint variables assume the same values at the initial and final points, because of their periodicity, and, using the boundary conditions, one deduces $\lambda_{\phi f} = 0$, which determines $C_2 = 0$ ($C_2 = \pi$ would produce a symmetrical trajectory with $\Omega_f = 180$ deg). Moreover, one easily recognizes that $K_2 = 0$ (that is, the acceleration in the radial direction is null, $T_u = 0$, or, equivalently, $\alpha = 0$), and that C_1 is indeterminate. The optimal thrust angle is then obtained from

$$\tan\beta = T_w/T_v = [K_3/(2K_1)]\cos\vartheta \tag{37}$$

in agreement with the solution assumed by Edelbaum. Note that this solution only holds when a complete revolution is performed. The value for the constant K_3/K_1 is determined by the required inclination change.

The choice of a different value for Ω_f would simply require different boundary conditions; because of the small inclination, $\phi_f = -\bar{i}\sin\Omega_f$ and $w_f = \bar{i}\cos\Omega_f$ replace $\phi_f = 0$ and $w_f = \bar{i}$. These different conditions modify the value of the integration constant, which becomes $C_2 = \Omega_f$, thus maintaining the relation between the thrust angle and the angular distance from the node. The results of the integration over one revolution do not change.

The suboptimal control law with constant-thrust angle β is adopted to obtain a closed-form solution (more precisely, the absolute value of β is constant, and the thrust angle is kept positive during the 180 deg arc centered around the ascending node, and negative during the other half-revolution; for instance, β is switched to negative values at $\vartheta = 90$ deg and back to positive values at $\vartheta = 270$ deg to obtain $\Omega_f = 0$). The equations of motion are readily integrated over one revolution to obtain the variation of radius and inclination; the transfer time and the mass variation, which is negligible over one revolution but will be important for multirevolution trajectories, are also computed. A constant exhaust velocity $c = c_N$ is assumed, and the thrust power $P = T_N c_N/2$ is introduced to obtain

$$\Delta r = 4\pi T_N \cos\beta = 8\pi P \cos\beta / c_N \tag{38}$$

$$\Delta i = 4T_N \sin\beta = 8P \sin\beta / c_N \tag{39}$$

$$\Delta t = 2\pi \tag{40}$$

$$\Delta m = -2\pi T_N / c_N = -4\pi P / c_N^2 \tag{41}$$

C. Variable-Specific-Impulse Thruster

The additional control c is obtained by setting the partial derivative of Eq. (24) with respect to the same control to zero. One easily obtains

$$c = 2\lambda_m/\Lambda = 1/\Lambda = 1/\sqrt{4K_1^2 + K_3^2\cos^2\vartheta} \tag{42}$$

as $K_2 = 0$ and $C_2 = 0$ still hold, and $\Lambda^2 = 4K_1^2 + K_3^2\cos^2\vartheta$. The equations of motion are now integrated over one revolution with the exact optimal control law ($T_u = 0$, $T_v = 4PK_1$, i.e., the thrust component parallel to the spacecraft velocity is constant, $T_w = 2PK_3\cos\vartheta$), to obtain

$$\Delta r = 16\pi P K_1 \tag{43}$$

$$\Delta i = 2\pi P K_3 \tag{44}$$

$$\Delta t = 2\pi \tag{45}$$

$$\Delta m = -2\pi P\left(8K_1^2 + K_3^2\right) \tag{46}$$

The parameters K_1 and K_3 are determined to accomplish the prescribed change in mass and inclination. The planar case ($\Delta i = 0$ and $K_3 = 0$) prescribes constant specific impulse and thrust, and provides the same results as in the preceding subsection for $K_1 = 0.5/c_N$, which is obtained from Eqs. (41) and (46) for the same propellant consumption Δm. On the other hand, the pure plane change maneuver ($\Delta r = 0$) requires $K_1 = 0$ and the exhaust velocity, provided by Eq. (42), varies according to $c = 1/|K_3\cos\vartheta|$. The inclination change can be compared with that of the constant-thrust case for the same thrust power and propellant consumption; from Eqs. (41) and (46), one gets $K_3 = \sqrt{2}/c_N = T_N/(\sqrt{2}P)$, which is inserted in Eq. (44) and provides $\Delta i = \sqrt{2}\pi T_N \approx 4.44 T_N$, which is larger than the value $4T_N$ of the constant-thrust case.

Figure 1 compares the results obtained in this subsection to Edelbaum's results (Sec. III.B), by presenting the radius increment as a function of the plane rotation, assuming the same propellant consumption. In particular, two control strategies with constant exhaust velocity are considered: the optimal steering law (variable β) and the suboptimal strategy employing a constant-thrust angle β, whose value can be directly read in Fig. 1 as $\tan\beta = \pi\Delta i/\Delta r$. The constant-$\beta$ operation implies a constant out-of-plane component of the thrust, which is useful to change the inclination only in the proximity of the nodes, but is wasted elsewhere. The thrust vector control improves the performance compared with the constant-β maneuver; the difference is greater for β close to 45 deg, whereas it vanishes for the coplanar transfer and the simple plane rotation. The third strategy exploits the capability of controlling both the thrust direction and magnitude. The additional control of the thrust magnitude allows an improved management of the available propellant, by using large thrust at the nodes and by increasing the specific impulse where the thrust cannot be usefully exploited to rotate the orbit plane. The improvement provided by the variation of the thrust magnitude is small for moderate plane rotations, but is significant when a large inclination change is sought and the thrust vector control provides modest benefits with respect to the constant-β maneuver. The larger inclination change of the variable-thrust case is obtained with total impulse $I_t = \int T\mathrm{d}t$, which is lower than in the constant-thrust case (e.g., $4\sqrt{2}T_N$ vs $2\pi T_N$ for $\Delta r = 0$), because a large thrust is used only near the nodes ($\vartheta = 0$ deg), where it is effective to the plane change, but the thrust is reduced elsewhere.

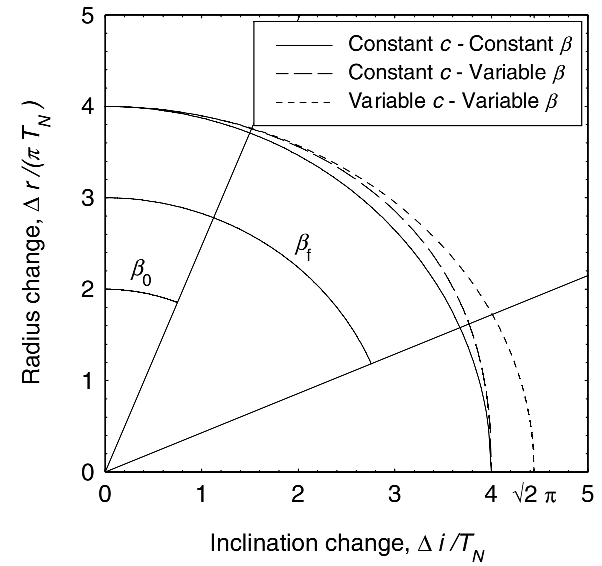

Fig. 1 Radius increment and plane rotation achievable using different control strategies with the same propellant consumption.

IV. Multirevolution Transfer

Multirevolution trajectories between inclined circular orbits are considered in the present section, focusing the attention on an LEO–GEO transfer. The optimal trajectory is obtained by combining a large number of one-revolution trajectories with small radius and inclination increments. The solutions of the preceding section have been obtained, and are immediately applicable, when dealing with the first revolution ($r = 1$, $m = 1$). During the maneuver, the spacecraft moves farther from the Earth and its mass diminishes because of the propellant consumption. Equations (38–41) and (43–46) must be rescaled to take the radius increment and the mass reduction into account. In particular, the radius, time, and mass variations on the left-hand side are divided by r, $\sqrt{r^3}$, and m, respectively; the thrust, exhaust velocity, and power on the right-hand side are divided by m/r^2, $\sqrt{1/r}$, and $m/\sqrt{r^5}$, respectively.

When multiple revolutions are considered, the relevant equations are therefore rewritten in the form

$$\Delta r = 4\pi r^3 \cos\beta T/m \tag{47}$$

$$\Delta i = 4r^2 \sin\beta T/m \tag{48}$$

$$\Delta t = 2\pi\sqrt{r^3} \tag{49}$$

$$\Delta m = -2\pi\sqrt{r^3}T/c \tag{50}$$

if both the thrust direction β and the exhaust velocity c are kept constant during each revolution, and

$$\Delta r = 16\pi P r^3 K_1/m \tag{51}$$

$$\Delta i = 2\pi P r^2 K_3/m \tag{52}$$

$$\Delta t = 2\pi\sqrt{r^3} \tag{53}$$

$$\Delta m = -2\pi P\sqrt{r^3}\left(8K_1^2 + K_3^2\right) \tag{54}$$

if the steering angle and the specific impulse are continuously adjusted according to Eqs. (37) and (42). One should note that $1/K_1$ and $1/K_3$ are velocities, and K_1 and K_3 have therefore been divided by $\sqrt{r}$.

Discrete changes are used to approximate a continuous variation of the relevant variables. The radius is assumed as the independent variable and the derivative of the generic variable $\mathrm{d}x/\mathrm{d}r$ is approximated by $\Delta x/\Delta r$.

A. Constant Specific Impulse and Mass

The constancy of the spacecraft mass implies infinite specific impulse and Eq. (50) can be neglected. By combining Eqs. (47–49) one obtains ($m = 1$)

$$\mathrm{d}\,i/\mathrm{d}r = \tan\beta/(\pi r) \tag{55}$$

$$\mathrm{d}\,t/\mathrm{d}r = m/(2\sqrt{r^3}T\cos\beta) \tag{56}$$

which are equivalent to Eqs. (7) and (8), as $\mathrm{d}V/V = -\mathrm{d}r/(2r)$ for quasi-circular trajectories.

The thrust angle β is the control variable. Without loss of generality, the initial orbit is fixed on the equatorial plane. The boundary conditions prescribe $i_0 = 0$ and $t_0 = 0$ at the initial point ($r_0 = 1$), and $i_f = \bar{i}$ at the final point ($r_f = \bar{r}$). The Hamiltonian is defined as

$$H = \lambda_i \tan\beta/(\pi r) + \lambda_t m/(2\sqrt{r^3}T\cos\beta) \tag{57}$$

and one easily deduces that the adjoint variables are constant ($\mathrm{d}\lambda_i/\mathrm{d}r = \mathrm{d}\lambda_t/\mathrm{d}r = 0$), and the optimal steering law is

$$\sin\beta = -\frac{2\lambda_i T}{\pi\lambda_t m}\sqrt{r} = K\sqrt{r} = K/V \tag{58}$$

which obviously coincides with Edelbaum's solution. The constant K is the only problem parameter and is numerically determined [15] to obtain the prescribed inclination when the final radius is reached, according to Eq. (55), which becomes

$$\mathrm{d}\,i/\mathrm{d}r = K/(\pi\sqrt{r}\sqrt{1-K^2 r}) \tag{59}$$

Equation (59) is integrated to obtain the algebraic equation

$$\pi i_f = \sin^{-1}(1-2K^2) - \sin^{-1}(1-2K^2 r_f) \tag{60}$$

which can be solved numerically for K.

B. Constant Specific Impulse and Variable Mass

The equation, which takes the mass variation into account, is added by combining Eqs. (38) and (41):

$$\mathrm{d}\,m/\mathrm{d}r = -m/(2c\sqrt{r^3}\cos\beta) \tag{61}$$

The Hamiltonian is now

$$H = \lambda_i \tan\beta/(\pi r) + \lambda_t m/(2\sqrt{r^3}T\cos\beta) - \lambda_m m/(2c\sqrt{r^3}\cos\beta) \tag{62}$$

and one again obtains $\mathrm{d}\lambda_i/\mathrm{d}r = \mathrm{d}\lambda_t/\mathrm{d}r = 0$ but

$$\mathrm{d}\,\lambda_m/\mathrm{d}r = (\lambda_m/c - \lambda_t/T)/(2\sqrt{r^3}\cos\beta) \tag{63}$$

The optimal steering law becomes

$$\sin\beta = \frac{2\lambda_i}{\pi m(\lambda_m/c - \lambda_t/T)}\sqrt{r} \tag{64}$$

The maximization of the final mass is considered (this problem is equivalent to the minimization of the transfer time, as the propellant flow rate is constant). The boundary conditions for optimality prescribe $\lambda_t = \lambda_{tf} = 0$ and $\lambda_{mf} = 1$; the latter is replaced by $\lambda_{m0} = 1$, because the problem is homogeneous in the adjoint variables. One easily recognizes that $m\lambda_m = 1$ is constant and again $\sin\beta = K/V$, in which $K = 2c\lambda_i/\pi$ is the parameter to be determined to obtain the prescribed plane change. Note that K assumes the same value for both constant and variable mass, if the same values of r_f and i_f are sought, because Eq. (60) holds in both cases.

C. Variable Specific Impulse and Mass

A constraint on the final time $t_f = \bar{t}$ must be imposed and added to the boundary conditions to obtain meaningful results, and λ_{tf} is now free. However, $\lambda_{t0} = \lambda_{tf} = -1$ replaces $\lambda_{m0} = 1$ to scale the adjoint variables. Two different strategies of increasing complexity can be envisaged for the thrust power management.

1) Exhaust velocity and thrust magnitude are kept constant during each revolution but are varied as the trajectory proceeds. Equations (55), (56), and (61) and the Hamiltonian Eq. (62) are still valid, but c is an additional control variable and $T = 2P/c$. The optimal controls are then obtained; $\partial H/\partial c = 0$ provides $c = \sqrt{2P\lambda_m}$ (which determines $T = \sqrt{2P/\lambda_m}$), whereas

$$\sin\beta = \frac{2\lambda_i}{\pi m(\lambda_m/c + 1/T)}\sqrt{r} = \frac{\sqrt{2P}\lambda_i}{\pi m\sqrt{\lambda_m}}\sqrt{r} \tag{65}$$

The two parameters λ_i and λ_{m0} are numerically determined to obtain the prescribed plane change and trip time. In this case $m^2\lambda_m = \lambda_{m0}$ is found to be constant, as it usually occurs when variable specific impulse is considered [5]. The specific impulse is therefore proportional to $\sqrt{P}/m$ and the thrust to $m\sqrt{P}$ (that is, the acceleration is constant), and Eq. (65) is again equivalent to $\sin\beta = K/V$, in which $K = \sqrt{2P/\lambda_{mo}}\lambda_i/\pi$ assumes the same value of the previous cases, if the same r_f and i_f are sought.

2) The specific impulse can be varied also during each revolution. By using Eqs. (51–54) one obtains

$$\mathrm{d}\,i/\mathrm{d}r = K_3/(8rK_1) \tag{66}$$

$$\mathrm{d}\,t/\mathrm{d}r = m/(8P\sqrt{r^3}K_1) \tag{67}$$

$$\mathrm{d}\,m/\mathrm{d}r = -m\left(8K_1^2 + K_3^2\right)/(8\sqrt{r^3}K_1) \tag{68}$$

and the Hamiltonian is now

$$H = \lambda_i K_3/(8rK_1) + \lambda_t m/(8P\sqrt{r^3}K_1) - \lambda_m m\left(8K_1^2 + K_3^2\right)/(8\sqrt{r^3}K_1) \tag{69}$$

One again has $\mathrm{d}\lambda_i/\mathrm{d}r = \mathrm{d}\lambda_t/\mathrm{d}r = 0$, but

$$\frac{\mathrm{d}\lambda_m}{\mathrm{d}r} = \frac{-\lambda_t/P + \lambda_m\left(8K_1^2 + K_3^2\right)}{8\sqrt{r^3}K_1} \tag{70}$$

The optimal control law is obtained by enforcing $\partial H/\partial K_1 = \partial H/\partial K_3 = 0$, which are combined to provide

$$K_3 = \frac{\lambda_i\sqrt{r}}{2m\lambda_m} \tag{71}$$

$$K_1 = \sqrt{\frac{1}{8}\left(\frac{1}{\lambda_m P} - K_3^2\right)} \tag{72}$$

The parameters λ_i and λ_{m0} are found by integrating Eqs. (66–68) and (70) to obtain the required trip time and plane change. The instantaneous values of the thrust angle and exhaust velocity, depending on radius and longitude, are expressed by Eqs. (37) and (42).

V. Results

The transfer from a 7000 km LEO to the geostationary orbit is considered; the plane change is fixed at 28.5 deg. The initial thrust acceleration is $T_N/m_0 = 0.35\ \mathrm{mm/s^2}$. Edelbaum's solution ($c = \infty$) is first compared with transfers that use the same suboptimal control law but different constant values of the specific impulse, which are typical of available electric propulsion systems. The results presented in Table 1 show that the trip time and the number of revolutions remarkably decrease with the specific impulse. A larger acceleration is obtained when the spacecraft mass diminishes and the influence of the variable mass due to the propellant consumption cannot be neglected. Nevertheless, the velocity increment $\Delta V = g_0 I_s \ln(1/m_f)$ is constant and coincides with the value provided by Edelbaum's solution, i.e., 5.784 km/s, according to Eq. (9). The rocket equation could be used to determine the final mass, and the trip time would immediately follow.

A Hall thruster with nominal $I_{\mathrm{sp}} = 1500$ s, which corresponds to nondimensional $c_N = 1.95$, is considered throughout. The constant-thrust transfer (case B) is compared with the two strategies of thrust power management: the specific impulse is kept constant during a whole revolution but is changed in the following one, as the spacecraft mass diminishes (case C1); the specific impulse is continuously adjusted (case C2). Cases B and C1 assume the suboptimal control law (constant β during each revolution), whereas β varies continuously according to the optimal control law for case C2. The same trip time as that of case B (158.15 days) is enforced for cases C1 and C2.

Figures 2–4, and Table 2 summarize the results of the analysis. Figure 2 compares the thrust angle histories; β varies as shown for cases B and C1 (the thrust angles β_0 and β_f during the initial and final revolution are shown in Fig. 1); β oscillates between $\pm\beta_{\max} = \pm\tan^{-1}[K_3/(2K_1)]$ during each revolution of case C2, according to Eq. (37). The inclination (Fig. 3) grows more rapidly in the initial part of the trajectory C2 with respect to cases B and C1, whereas the opposite occurs in the final phases. Figure 4 compares the exhaust velocities; c varies between $c_{\min} = 1/\sqrt{4K_1^2 + K_3^2}$ and $c_{\max} = 1/(2K_1)$ during each half-revolution of case C2, with the average value

$$c_{\mathrm{avg}} = \frac{\int_0^{2\pi} T\,\mathrm{d}\vartheta}{\int_0^{2\pi} (T/c)\,\mathrm{d}\vartheta} = \frac{\int_0^{2\pi} (2P/c)\,\mathrm{d}\vartheta}{\int_0^{2\pi} (2P/c^2)\,\mathrm{d}\vartheta} = \frac{\int_0^{2\pi} (1/c)\,\mathrm{d}\vartheta}{\int_0^{2\pi} (1/c^2)\,\mathrm{d}\vartheta} \tag{73}$$

that closely follows the exhaust velocity of case C1. In comparison with case B, both the variable-I_{sp} trajectories use a lower specific impulse, i.e., a larger thrust, in the initial phases of the transfer, whereas c grows and T diminishes in the final phases. The larger initial thrust provides a faster increment of the orbit radius and each revolution requires more time. Therefore, the number of revolutions of cases C1 and C2 is lower than in case B. In the final phases of the mission, $c_{\max}$ grows to very large values; however, in these circumstances the thruster could be turned off without changing the trajectory appreciably, as the corresponding thrust is very low. Table 2 also shows the overall velocity increment

Table 1 Performance of LEO–GEO transfers using Edelbaum's suboptimal control law and constant-thrust magnitude

Type	Specific impulse, s	Trip time, days	Revolutions	Final mass
Edelbaum	∞	191	1048	1
Ion thruster	3000	174	989	0.822
Hall thruster	1500	158	936	0.675
Arcjet	600	122	802	0.374

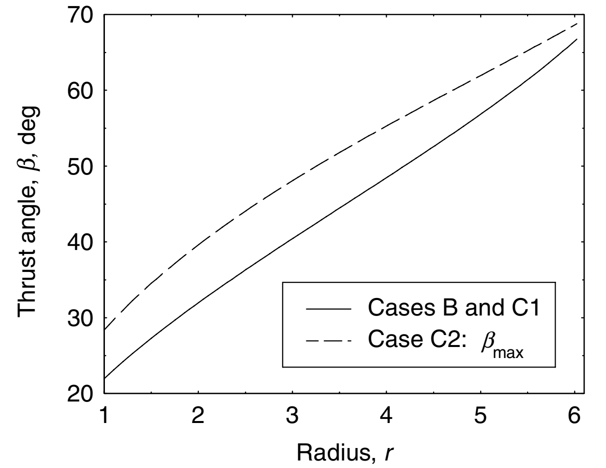

Fig. 2 Thrust angle as a function of radius for different control strategies.

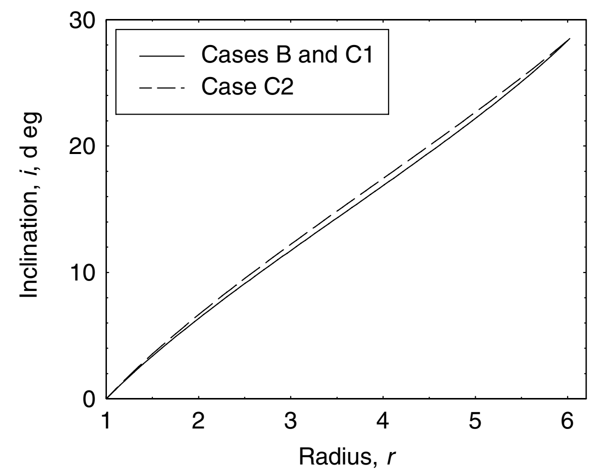

Fig. 3 Inclination as a function of radius for different control strategies.

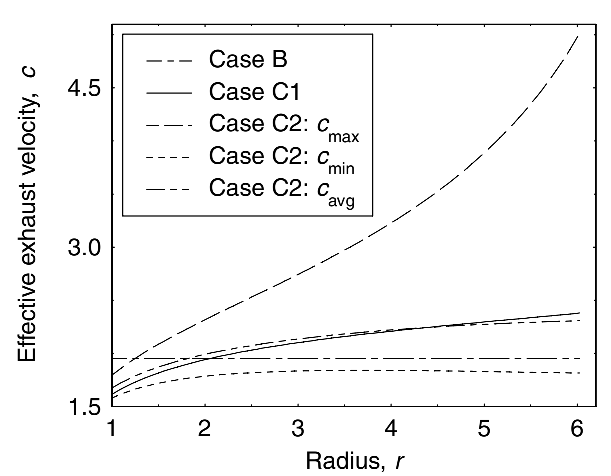

Fig. 4 Comparison of the exhaust velocities for different control strategies.

$$\Delta V = \int_0^{t_f} (T/m)\,\mathrm{d}t \tag{74}$$

and the average specific impulse of the whole mission, which is defined according to the rocket equation $(I_{\mathrm{sp}})_{\mathrm{avg}} = \Delta V/[g_0 \ln(1/m_f)]$. The improvement in terms of final mass, which is obtained in case C1, is only related to the management of the propellant in accordance with the variable spacecraft mass, which

Table 2 Trajectory performance using different control strategies

Case	Revolutions	ΔV, km/s	$(I_{sp})_{avg}$, s	Final mass
B	936	5.784	1500	0.6750
C1	867	5.784	1516	0.6778
C2	884	5.469	1527	0.6941

allows a larger $(I_{sp})_{avg}$. The benefit is significantly greater in case C2 and is mainly related to the ΔV reduction due to the control of both the thrust magnitude and direction, which improves the rotation of the orbit plane. The results are consistent with those presented by Kluever [16], who optimized the passage from geostationary transfer orbit to GEO by means of a direct method.

The control laws that have been used in this section had been derived assuming quasi-circular orbits and neglecting the mass variation over one revolution. Errors arise when these laws are used to integrate the exact equations of motion [2–5]. However, the integration constants are two available parameters that can be redetermined to fulfill two boundary conditions which prescribe the final radius and inclination, namely $r_f = \bar{r}$ and $\cos \bar{i} = \cos \phi_f \sin(w_f / \sqrt{w_f^2 + v_f^2})$ at $t = t_f$. The final eccentricity cannot be set to zero, but it is always less than 0.1%; the longitude of the ascending node, which actually does not matter on arrival to GEO, is also lower than 0.1 deg. The resulting differences in the control laws are negligible; the thrust angle and the specific impulse differ always less than 0.01 deg and 1 s, respectively, at the same distance from the main body. The final mass obtained by integrating the exact equations of motion is roughly 0.01% larger than using the approximate equations.

A final step has been carried out to testify to the accuracy of the whole procedure. Using concepts and hints provided by the present analysis, the authors succeeded in obtaining the numerical solution of the rigorous optimization problem with either constant or variable specific impulse (corresponding to cases B and C2, respectively). The indirect optimization method is the same as the one that was adopted for interplanetary missions [2–5], but uses the true longitude ϑ as the independent variable. The final mass of the exact solution presents a noticeable increment (0.6819 vs 0.6750) when constant specific impulse is considered, because the constant-β assumption of the approximate solution has been removed; the other features of the trajectories do not change, except the number of revolutions which grows to 938. On the other hand, when variable specific impulse is adopted, the final mass is almost the same (0.6942 vs 0.6941) but the number of revolutions has a significant growth to 899.

VI. Conclusions

The authors have developed a simple and easily implemented method, which is based on the extension of Edelbaum's approach, for the solution of the multirevolution transfer from low Earth orbit to geostationary orbit with very low thrust, which is usually very difficult to be obtained with traditional optimization methods; only one or two parameters (depending on the problem) define the optimal trajectory, and can be easily determined with numerical methods. Edelbaum's analysis of the low-thrust transfer between noncoplanar circular orbits has been first extended to take the mass variation into account; the reduction of the spacecraft mass, which is due to the propellant consumption, has a remarkable effect on the trip time of the multirevolution transfer from a low Earth orbit with 28.5 deg inclination to the equatorial geostationary orbit. The mass of the exhausted propellant is significant and improved performance is provided by variable-specific-impulse propulsion, even though Edelbaum's suboptimal steering law (that is, constant-thrust direction during each half-revolution around the main body) is maintained. The characteristic ΔV is the same as for operation with constant nominal thrust and exhaust velocity, but the propellant consumption is reduced by exploiting the available power with more thrust and lower exhaust velocity at the beginning of the transfer, and vice versa at the end of the mission.

A more important result is that the availability of variable-specific-impulse thrusters allows an improved use of the thrust during the basic one-revolution maneuver. A more efficient plane rotation is obtained by controlling not only the thrust direction but also its magnitude, which is purposely increased in the proximity of the nodes. The thrust vector control is more useful when the plane rotation is modest, in comparison with the radius increment, but becomes less efficient when the plane rotation is predominant; in this case, the control of the exhaust velocity is almost mandatory for improved performance. From a theoretical point of view, if the specific impulse is an additional control, an analytical solution of the optimal single-revolution transfer can be obtained and used to carry out an approximate but very accurate analysis of the multirevolution transfer.

The theoretical analysis of the one-revolution transfer has also provided the authors with useful suggestions to obtain the convergence of their indirect numerical procedure to the solution of the complete Edelbaum's problem in both cases of constant and variable specific impulse. The exact solution of the former case shows that the variation of the specific impulse and thrust magnitude during each revolution is more effective than the rotation of the thrust to improve the complete transfer from low to geostationary orbit. The exact numerical solution of the variable-specific-impulse problem proves the accuracy of the approximate solution outlined in this paper.

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